Tsirelson bounds for quantum correlations with indefinite causal order

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Tsirelson bounds in Bell nonlocality

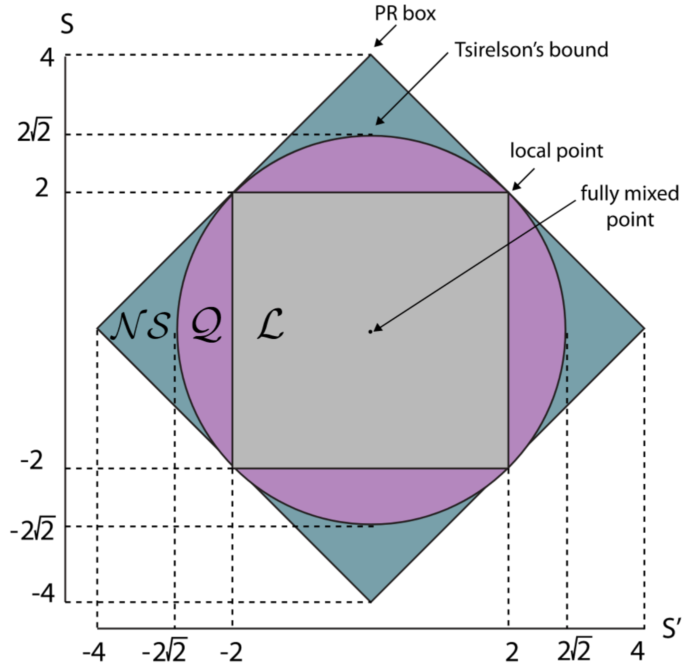
- quantum correlations between distant events:

$$p(a,b|x,y) = \langle\psi|M_{a|x}N_{b|y}|\psi\rangle$$

- CHSH inequality:

$$\mathcal{I}_{\mathrm{CHSH}} := \sum_{a,b,x,y} (-1)^{a+b+xy} p(a,b|x,y) \leq 2$$

- **Tsirelson's bound**: $\mathcal{I}_{\mathrm{CHSH}} = 2\sqrt{2}$ is the maximal CHSH correlation permitted by quantum mechanics;
- general method: Navascues-Pironio-Acin (NPA) hierarchy.



Brunner, Cavalcanti, Pironio, Scarani & Wehner (2014)

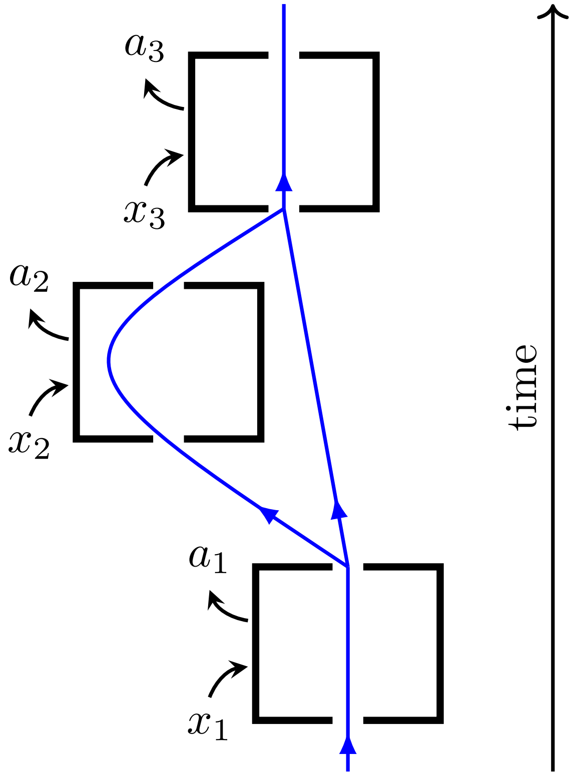
Bell-like inequalities for causality

- free choice
- closed laboratories
- global causal structure

the observed statistics is a **causal probability distribution**:

$$p(a_1, \cdots, a_N \mid x_1, \cdots, x_N)$$

Oreshkov, Costa & Brukner (2012)



- settings $\vec{x} = (x_1, \cdots, x_N)$
- outcomes $\vec{a} = (a_1, \cdots, a_N)$
- real coefficients $(\alpha_{\vec{a},\vec{x}})$
- an N-party correlation

$$\mathcal{J} = \sum_{\vec{a},\vec{x}} \alpha_{\vec{a},\vec{x}}\, p(\vec{a} \mid \vec{x})$$

Causal inequalities

$$p(\vec{a} \mid \vec{x}) \text{ is causal} \implies \mathcal{J} \leq \beta$$

Guess Your Neighbor's Input (GYNI)

- GYNI game [Almeida *et al* (2010)]: Alice/Bob generates a random bit x_1/x_2, the goal is that each player makes a correct guess of the other's bit.

- two-party causal probability distribution: random mixture of

  $B \not\preceq A \quad p_A(a_1 \mid x_1, x_2) = p_A(a_1 \mid x_1, x_2'), \forall x_1, x_2, x_2', a_1$

  $A \not\preceq B \quad p_B(a_2 \mid x_1, x_2) = p_B(a_2 \mid x_1', x_2), \forall x_1, x_2, x_1', a_2.$

- Causal probability distributions form a **convex polytope**.

Branciard, Araújo, Feix, Costa and Brukner (2016)

Oreshkov and Giarmatzi (2016)

AAbbott, Giarmatzi, Costa & Branciard (2016)

- GYNI correlation

$$\mathcal{J}_{\text{GYNI}} := \Pr(a_1 = x_2, a_2 = x_1)$$

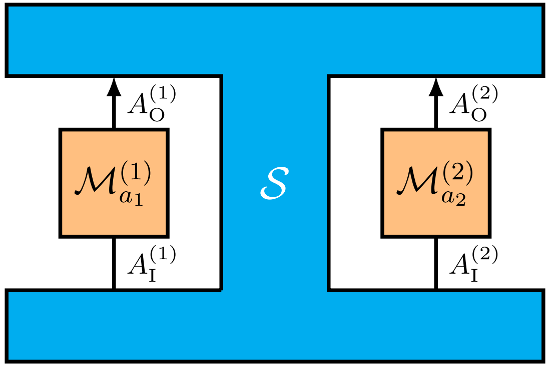
causal inequality

$$\mathcal{J}_{\text{GYNI}} \leq \frac{1}{2}$$

Local quantum theory and process matrices

- drop the assumption of a global causal structure;
- keep logical consistency with local quantum operations:

$$p(\vec{a}) = \mathcal{S}\left(\mathcal{M}_{a_1}^{(1)}, \cdots, \mathcal{M}_{a_N}^{(N)}\right)$$



Chiribella, D'Ariano, Perinotti & Valiron (2009)
Oreshkov, Costa & Brukner (2012)

Choi-Jamiołkowski isomorphism:

$$\mathcal{M}_{a_i}^{(i)} \iff M_{a_i}^{(i)}$$

$$\mathcal{S} \iff S \quad \text{(process matrix)}$$

$$p(\vec{a}) = \mathrm{Tr}\left[S^{\mathrm{T}} \bigotimes_{i=1}^{N} M_{a_i}^{(i)}\right]$$

Process matrices: $S \geq 0$; S in the dual affine set of Choi operators of no-signalling channels.

Quantum correlations with Indefinite Causal Order

Violations of causal inequalities

- Branciard *et al* (2016): numerical value calculated with a process matrix on 5-dimensional quantum systems

$$\mathcal{J}_{\mathrm{GYNI}} = 0.6218 > \frac{1}{2}$$

- Kunjwal and Oreshkov (2023): no perfect win for GYNI.

What are the maximum violations (ICO bounds) of causal inequalities?

Research gap

- Brukner (2015): maximum OCB correlation [Oreshkov, Costa & Brukner (2012)] with a restricted set of local quantum operations;
- Branciard *et al* (2016): see-saw optimization for correlations realized in fixed-dimensional systems;
- Bavaresco *et al* (2019): constraints for GYNI correlation realized in finite-dimensional systems (trivial for infinite dimensional systems);

The optimization problem for ICO bounds

$$\mathcal{J}^{\mathrm{ICO}} = \sup_{\mathcal{S}} \sup_{\left(\mathcal{M}^{(i)}_{a_i|x_i}\right)} \mathcal{S}\left(\sum_{\vec{a},\vec{x}} \alpha_{\vec{a},\vec{x}} \bigotimes_{i=1}^{N} \mathcal{M}^{(i)}_{a_i|x_i}\right)$$

- $\mathcal{S}$: quantum process (process matrix);
- $\mathcal{M}^{(i)}_{a_i|x_i}$: instruments;
- $\alpha_{\vec{a},\vec{x}}$: coefficients of the correlation function

$$\mathcal{J} = \sum_{\vec{a},\vec{x}} \alpha_{\vec{a},\vec{x}}\, p(\vec{a} \mid \vec{x})$$

Challenges

1. dimensions of quantum systems not fixed;
2. complex affine constraints for process matrices.

Our method: single-trigger SDP relaxation

- For **single-trigger** correlations, ICO bounds can be saturated by a **canonical choice of instrument**

$$\mathcal{J}^{\mathrm{ICO}}_{\text{single-trigger}} = \max_{\mathcal{S}} \mathcal{S}\left(\sum_{\vec{a},\vec{x}} \alpha_{\vec{a},\vec{x}} \bigotimes_{i=1}^{N} \mathcal{M}^{(i)\star}_{a_i|x_i}\right)$$

- For an arbitrary correlation $\mathcal{J}$, decompose it into a sum of single-trigger correlations:

$$\mathcal{J} = \sum_j \mathcal{J}_j$$

- The optimal upper bound can be expressed as a **semidefinite program (SDP)**.

SDP relaxation

$$\mathcal{J} \leq \min_{\{\mathcal{J}_j\}} \left\{ \sum_j \mathcal{J}^{\mathrm{ICO}}_j \right\}$$

Single-trigger correlations

A correlation $\mathcal{J} = \sum_{\vec{a},\vec{x}} \alpha_{\vec{a},\vec{x}}\, p(\vec{a} \mid \vec{x})$

is **single-trigger** if

for every party i, there exists **a special setting (the trigger)** ξ_i such that

$$x_i \neq \xi_i \implies \alpha_{\vec{a},\vec{x}} \text{ is independent of } a_i$$

Two-party case

$$\forall a_1, a_1', a_2, a_2', x_1, x_2$$

$$x_1 \neq \xi_1 \implies \alpha_{a_1,a_2,x_1,x_2} = \alpha_{a_1',a_2,x_1,x_2}$$

$$x_2 \neq \xi_2 \implies \alpha_{a_1,a_2,x_1,x_2} = \alpha_{a_1,a_2',x_1,x_2}$$

Example: Lazy Guess Your Neighbor's Input (LGYNI)

- [Branciard et al (2016)] Alice/Bob generates a random bit x_1/x_2, and they guess the other player's setting only when their own setting is 1:

$$\mathcal{J}_{\mathrm{LGYNI}} := \Pr(x_1(a_1 \oplus x_2) = 0, x_2(a_2 \oplus x_1) = 0).$$

- causal inequality: $\mathcal{J}_{\mathrm{LGYNI}} \leq \frac{3}{4}$.
- triggers: $\xi_1 = \xi_2 = 1$.

$$\begin{aligned} &\Pr(x_1(a_1 \oplus x_2) = 0, x_2(a_2 \oplus x_1) = 0 \mid x_1 = 0, x_2) \\ =\ &\Pr(x_2(a_2 \oplus x_1) = 0 \mid x_1 = 0, x_2) \end{aligned}$$

Canonical instrument ⋆

The canonical local operations, with $\xi_1, \xi_2, ..., \xi_N$ being the triggers

- Encode the setting in an auxiliary system:

$$\mathcal{M}^{(i)\star}_{a_i|x_i} = \mathcal{N}^{(i)\star}_{a_i|x_i} \otimes |x_i\rangle\langle x_i|_{\text{aux}}$$

- If $x_i = \xi_i$, measure in computational basis, otherwise do not perform measurement and generate a random outcome:

$$\mathcal{N}^{(i)\star}_{a_i|x_i}(\rho) := \begin{cases} |a_i\rangle\langle a_i|\rho|a_i\rangle\langle a_i| \text{ if } x_i = \xi_i \\ \frac{1}{m_i}\rho \qquad \text{otherwise (random outcome)} \end{cases}$$

Explicit formulas

- performance operator $\Omega^\star := \sum_{\vec{a},\vec{x}} \alpha_{\vec{a},\vec{x}} \bigotimes_{i=1}^{N} M^{(i)\star}_{a_i|x_i}$
- in terms of **max relative entropy distance** (w.l.o.g., assume $\Omega^\star \geq 0$)

$$\mathcal{J}^{\text{ICO}}_{\text{single-trigger}} = 2^{D_{\max}(\Omega^\star \,\|\, \text{NoSig})}$$

- expressed through **SDP**:

$$\begin{aligned} \text{maximize} \quad & \operatorname{Tr}(S\Omega^\star) \\ \text{subject to} \quad & S \in \text{DualAff}(\text{NoSig}) \\ & S \geq 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & \eta \\ \text{subject to} \quad & C \in \text{Aff}(\text{NoSig}) \\ & \eta C \geq \Omega^\star \end{aligned}$$

Sketch of the proof of single-trigger ICO bounds

$\mathcal{I}(\mathbb{M})$: the range of a correlation $\mathcal{I}$ generated by arbitrary process matrices and a restricted set $\mathbb{M}$ of instruments.

$\mathbb{M}_1 := \{\text{all instruments}\}$

$\mathbb{M}_2 := \{\text{labelled projective instruments}\}$

$\mathbb{M}_3 := \{\text{single-trigger instruments}\}$

$\mathbb{M}_4 := \{\text{the canonical instrument}\}$

Step 1 $\mathcal{I}(\mathbb{M}_1) = \mathcal{I}(\mathbb{M}_2)$;

Step 2 (single-trigger)

$\max \mathcal{I}(\mathbb{M}_2) = \max \mathcal{I}(\mathbb{M}_3)$.

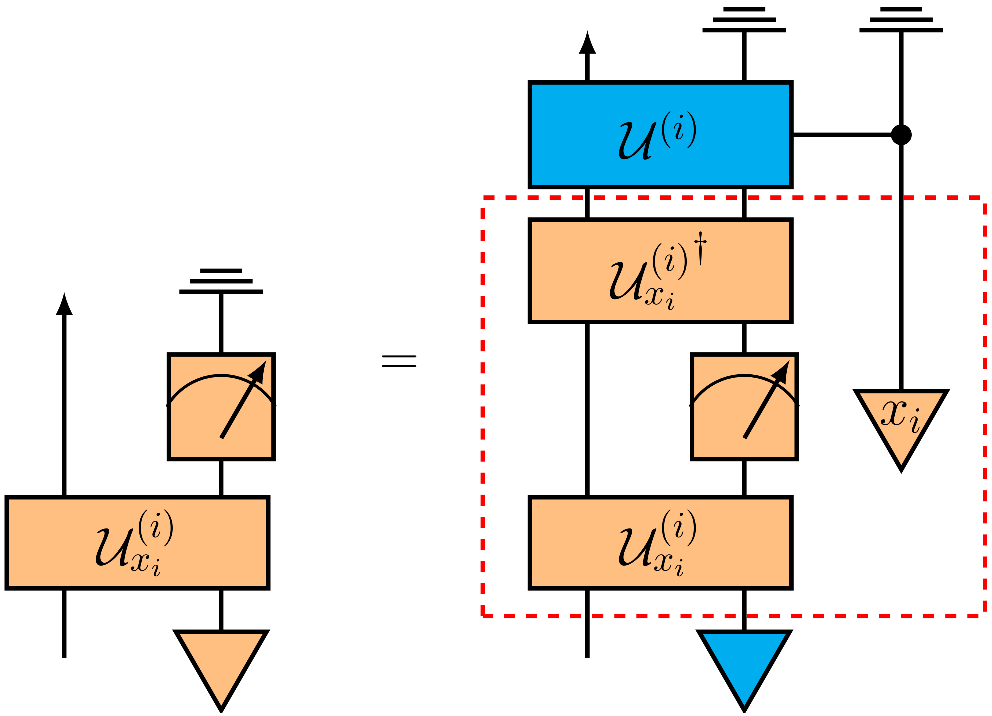
Step 3 $\mathcal{I}(\mathbb{M}_3) = \mathcal{I}(\mathbb{M}_4)$.

Step 1: reduction to labelled projective instruments

- unitary realization (Ozawa's dilation theorem)
- labelled projective instruments:

$$\mathcal{P}^{(i)}_{a_i|x_i} \otimes |x_i\rangle\langle x_i|_{\text{aux}}$$

where $\left(\mathcal{P}^{(i)}_{a_i|x_i}\right)_{a_i}$ is an **equal-rank** projective measurement for every setting.



Step 2: reduction to single-trigger instruments

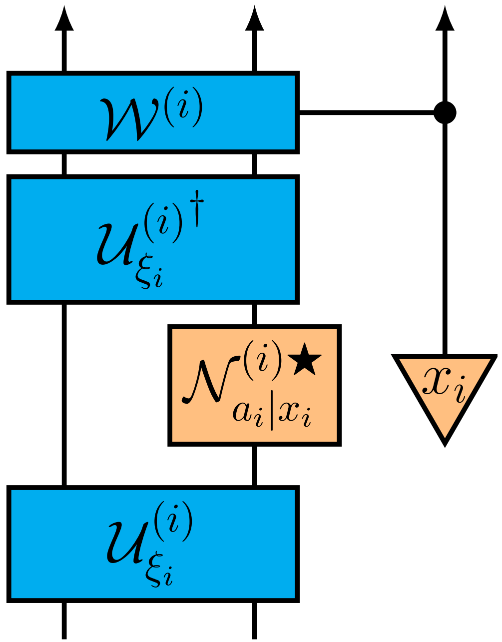
- **single-trigger instruments** (unitary for non-trigger settings):

$$\mathcal{M}^{(i)}_{a_i|x_i}(\rho) = \begin{cases} P^{(i)}_{a_i|\xi_i}\rho P^{(i)}_{a_i|\xi_i} \otimes |\xi_i\rangle\langle\xi_i|_{\text{aux}} & \text{if } x_i = \xi_i \\ \frac{1}{m_i} W^{(i)}_{x_i}\rho W^{(i)\dagger}_{x_i} \otimes |x_i\rangle\langle x_i|_{\text{aux}} & \text{otherwise} \end{cases}$$

- the value of a single-trigger correlation is unchanged when **discarding the outcomes** for non-trigger settings;
- a projective measurement followed by discarding the outcome is equivalent to a **random unitary channel**.

Step 3: reduction to the canonical instrument

- Single-trigger instruments can be obtained from the canonical instrument locally.



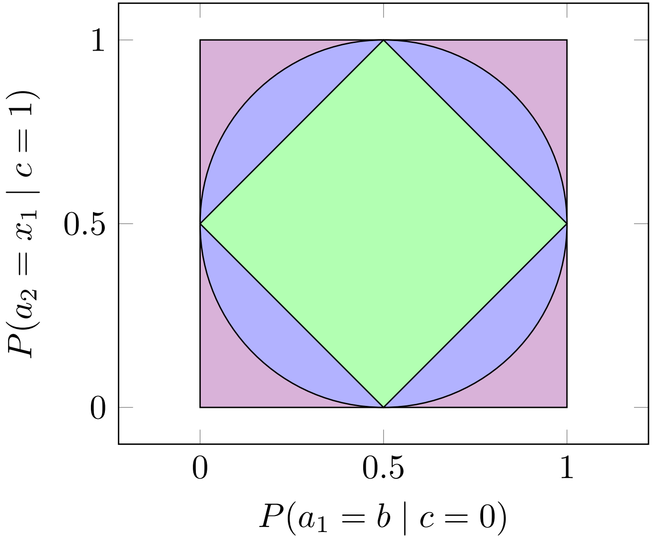
Upper bounding GYNI and LGYNI correlations

$$\mathcal{J}_{\text{LGYNI}} = \frac{p(11|11) + p_A(0|10) + p_B(0|01) + 1}{4}$$
$$\leq 0.8194 \quad \text{(tight)}$$

$$\begin{aligned}
\mathcal{J}_{\text{GYNI}} \leq\ & \alpha_0 p(00|00) + \alpha_1 p_A(1|01) + \alpha_2 p_B(1|10) \ \text{triggers (0,0)} \\
& + \alpha_0 p(10|01) + \alpha_1 p_A(0|00) + \alpha_2 p_B(1|11) \ \text{triggers (0,1)} \\
& + \alpha_0 p(01|10) + \alpha_1 p_A(1|11) + \alpha_2 p_B(0|00) \ \text{triggers (1,0)} \\
& + \alpha_0 p(11|11) + \alpha_1 p_A(0|10) + \alpha_2 p_B(0|01) \ \text{triggers (1,1)} \\
\leq\ & 0.7592
\end{aligned}$$

A slice of the set of probability distributions with ICO

Probabilities in OCB correlation [Oreshkov, Costa & Brukner (2012)], where Bob's setting is a pair of bits (b, c).



- **Causal probability distributions** form a convex polytope (the inner square);
- **Probability distributions realized by ICO processes** does NOT form a polytope (the circle)
- Every **unconstrained probability distribution** (outer square) can be realized in bistochastic quantum theory [Chiribella and Liu (2022)].

Potential improvements of our method

Better method?

Is there **a hierarchy of SDP relaxations** (similar to the NPA hierarchy in Bell nonlocality) for the ICO-bound problem?

Broader method?

For computing Tsirelson bounds for correlations with **partially pre-defined causal structure;**

Including the following as special cases:

- no-signalling causal structure: NPA hierarchy
- no restriction on causal structure: single-trigger SDP

Deriving ICO bounds from physical principles?

Physical principles in the context of Bell inequalities:

- non-trivial communication complexity
- non-trivial nonlocal computation,
- information causality,
- macroscopic locality,
- local orthogonality.

van Dam, Ph.D. thesis (1999)
Brassard *et al* PRL (2006)
Brunner and Skrzypczyk PRL (2009)
Linden *et al* PRL (2007)
Pawlowski *et al* Nature (2009)
Navascués and Wunderlich (2010)
Fritz *et al* Nat. Comm. (2013)

Device-independent quantum cryptography with indefinite causality?

- self-testing process matrices and instruments;
- device-independent protocols beyond Bell scenarios, for example, DIQRNG under the assumption of signalling causal structures (partially pre-defined causal structure, no causal structure).

Take-home messages

1. Quantum theory is compatible with behaviors violating causal inequalities; **ICO bounds** of correlations represent a more complex problem than Tsirelson bounds in Bell nonlocality;
2. For single-trigger correlations, ICO bounds can be saturated by a **canonical choice of instruments**;
3. Single-trigger ICO bounds provide an **SDP relaxation** of the ICO-bound problem;
4. ICO bounds help to understand quantum theory and open up new possibilities of device-independent quantum cryptography.

Thank you for your attention!